ANSWER KEY

1. 6
2. 7.5
3. 2
4. 14
5. 18
6. $\frac{1}{2}(1+\sqrt{5})$
7. 15

1. It is relatively easy to compute values for $n = 1, 2, \ldots, 10$. To the nearest hundredth we obtain

$n =$	1	2	3	4	5	6	7	8	9	10
value	18.5	10	7.5	6.5	6.1	6	6.07	6.25	6.5	6.8

It appears that the minimal value is achieved at $n = \mathbf{6}$. This can be proved with the AM-GM inequality, which states that

$$\frac{n}{2} + \frac{18}{n} \leq 2\sqrt{\frac{n}{2} \cdot \frac{18}{n}} = 2\sqrt{9} = 6,$$

with equality if and only if $\frac{n}{2} = \frac{18}{n}$, which occurs at $n = 6$.

2. Moving the decimal point one place to the left is equivalent to dividing by ten. Therefore we seek the real number x for which $3 + \frac{1}{10}x = \frac{1}{2}x$. This equation becomes $3 = \frac{2}{5}x$, or $x = \frac{15}{2} = \mathbf{7.5}$. Sure enough, 3.75 is half of 7.5, as desired.

3. Perhaps the most obvious place to start is the upper left corner. The two-digit multiples of 7 that clearly will not work are 49, 70, 77, 91, and 98. We can also eliminate 63, since then the first vertical number would also have to be 63, but we can't use the 3 twice. This leaves 14, 21, 28, 35, 42, 56, and 84. Trying each of these cases goes pretty quickly, since the 9 is already in place. One finds that there are two possible solutions; one is shown at right and the other is obtained by swapping the 1 and the 8. Regardless, the circled digit must be a **2**.

5	6		
4	9	7	
	3	0	1
		(2)	8

4. There are three different types of rotations which return the cube to its original position in space. The first involves piercing the cube with an axis passing through the center of a pair of opposite faces, then spinning the cube by 90°, 180°, or 270°. But none of these are acceptable, since those opposite faces return to their starting positions. The second method is to pass an axis through a pair of opposite vertices, then spin the cube by 120° or 240°. There are four possible axes of this type, and two angles of rotation, giving eight such rotations, all of which work. Finally, one could take an axis through the midpoints of a pair of opposite edges and spin the cube by 180°; these six rotations are also good. Hence there are a total of **14** acceptable orientations.

5. We first notice that although the trinomial $m^2 + 7m + 89$ cannot be factored, $m^2 + 7m + 12 = (m+3)(m+4)$ can be. Since these two expressions differ by 77, the first will be divisible by 77 exactly when the second is. In order to be a multiple of 77, $(m+3)(m+4)$ must be a multiple of both 7 and 11. There are four possible ways to accomplish this. Two are obvious—make $(m+3)$ a multiple of 77 using $m = 74$, or make $(m+4)$ a multiple of 77 by taking $m = 73$. But it is possible to do better. For example, we can arrange for $7|(m+4)$ and $11|(m+3)$ using $m = 52$. And we find that $7|(m+3)$ and $11|(m+4)$ by choosing $m = \mathbf{18}$, the lowest possible answer.

6. This problem is not as hard as it looks. First, observe that there are positive real numbers x and y for which $x - y = 2009$ and $xy = 2009$. This occurs for

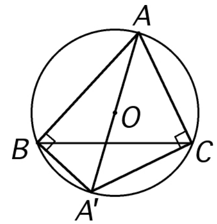
$$x = \tfrac{1}{2}\left(\sqrt{2009^2 + 4(2009)} + 2009\right) \quad y = \tfrac{1}{2}\left(\sqrt{2009^2 + 4(2009)} - 2009\right),$$

although the exact values are not important. Substituting these values for x and y into the given functional equation, we discover that

$$f(2009) = \sqrt{f(2009) + 1} \qquad \Longrightarrow \qquad f(2009)^2 - f(2009) - 1 = 0.$$

In other words, $f(2009)$ is the positive solution to $r^2 - r - 1 = 0$. This can be easily found via the quadratic formula to be $\mathbf{\frac{1}{2}(1+\sqrt{5})}$ which, of course, is the golden ratio.

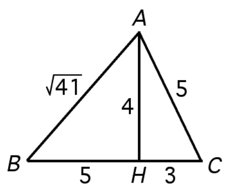
7. Let us label $BC = a$, $AC = b$, $AB = c$, $m\angle BAC = \alpha$, $m\angle ABC = \beta$, and $m\angle ACB = \gamma$. Then $m\angle BA'A = m\angle BCA = \gamma$, since these two angles subtend the same arc of the circle. Hence $BA' = c\cot\gamma$ using right triangle trig in $\triangle BA'A$. Similarly, $CA' = b\cot\beta$. Furthermore, $m\angle BA'C = \gamma + \beta = 180° - \alpha$. Thus we compute



$$\begin{aligned} area(BA'C) &= \tfrac{1}{2}(c\cot\gamma)(b\cot\beta)\sin(180° - \alpha) \\ &= area(BAC)\cdot\cot\beta\cot\gamma, \end{aligned}$$

because $area(BAC) = \frac{1}{2}bc\sin\alpha$ and $\sin\alpha = \sin(180° - \alpha)$.

There is a quick finish to this problem which may be found by drawing altitude $\overline{AH}$ to side $\overline{BC}$. It is not hard to determine that $AH = 4$, $BH = 5$, and $CH = 3$. (The details are left as an exercise to the reader.) We now easily compute that $area(ABC) = 16$, $\cot\beta = \frac{5}{4}$, and $\cot\gamma = \frac{3}{4}$. Using the above equation we conclude that $area(BA'C) = (16)(\frac{5}{4})(\frac{3}{4}) = \mathbf{15}$.



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