Team Play Solutions

Part i: Substituting in $b = 0$ we obtain $F_aF_0 + F_{a+1}F_1 = F_{a+1}$. Since $F_0 = 0$ and $F_1 = 1$, this reduces to just $F_{a+1} = F_{a+1}$, which is obviously true. Next, substituting $b = 1$ gives $F_aF_1 + F_{a+1}F_2 = F_{a+2}$. Since $F_2 = 1$ also, this time the relationship reduces to $F_a + F_{a+1} = F_{a+2}$, which is again clearly true by definition of the Fibonacci sequence.

When $a = 5$ and $b = 7$ we must confirm that $F_5F_7+F_6F_8 = F_{13}$. We first compute $F_5 = 5$, $F_6 = 8$, $F_7 = 13$ and $F_8 = 21$. Hence the left-hand side becomes $(5)(13) + (8)(21) = 65 + 168 = 233$. But $F_{13} = 233$, so the relationship holds in this case as well.

Part ii: We have seen that the relationship $F_aF_b + F_{a+1}F_{b+1} = F_{a+b+1}$ holds for $b = 0$ and $b = 1$. To prove that it holds in general, we assume that it is true for all values of b from $b = 0$ up to $b = k$, for some fixed number k. We will then argue that it is also true for $b = k + 1$, from which the result will follow by induction.

Since the statement is true for $b = k - 1$ and $b = k$, we know that

$$F_aF_{k-1} + F_{a+1}F_k = F_{a+k} \quad \text{and} \quad F_aF_k + F_{a+1}F_{k+1} = F_{a+k+1}$$

holds for all $a \geq 0$. Adding these two equalities yields

$$F_a(F_{k-1} + F_k) + F_{a+1}(F_k + F_{k+1}) = F_{a+k} + F_{a+k+1},$$

which simplifies to $F_aF_{k+1} + F_{a+1}F_{k+2} = F_{a+k+2}$. But this is exactly the case $b = k + 1$ which we wished to prove, so we are done.

Part iii: Replacing b by $a-1$ in the identity $F_aF_b+F_{a+1}F_{b+1} = F_{a+b+1}$ we find that $F_aF_{a-1}+F_{a+1}F_a = F_{2a}$. Factoring out an F_a on the left, we have $F_a(F_{a-1}+F_{a+1}) = F_{2a}$. Since the product of F_a and another integer (namely $F_{a-1}+F_{a+1}$) gives F_{2a}, we conclude that F_{2a} is a multiple of F_a.

We can use the same technique again by choosing $b = 2a-1$ instead. This time we obtain $F_aF_{2a-1} + F_{a+1}F_{2a} = F_{3a}$. We cannot immediately factor out an F_a as before, but we already know that F_{2a} is a multiple of F_a, hence the entire left-hand side must be also. More precisely, we could substitute our expression for F_{2a} from above, ultimately yielding

$$F_a(F_{2a-1} + F_{a-1}F_{a+1} + F_{a+1}^2) = F_{3a},$$

making it clear that F_{3a} is also a multiple of F_a.

Part iv: The first eight rows of the Fibonomial triangle look like

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 6 & & 3 & & 1 & & & \\
 & & 1 & & 5 & & 15 & & 15 & & 5 & & 1 & & \\
 & 1 & & 8 & & 40 & & 60 & & 40 & & 8 & & 1 & \\
1 & & 13 & & 104 & & 260 & & 260 & & 104 & & 13 & & 1
\end{array}$$

Part v: The identity $\left(\!\binom{n+1}{k+1}\!\right) = F_{n-k+1}\left(\!\binom{n}{k}\!\right) + F_k\left(\!\binom{n}{k+1}\!\right)$ is one of many marvelous relationships among the Fibonacci numbers. To prove it, we use the formula defining $\left(\!\binom{n}{k}\!\right)$, then find a common denominator. To save space, we have already cancelled all the factors from F_{n-k} down to F_1 in the expression for $\left(\!\binom{n}{k}\!\right)$. We find that

$$\begin{aligned}
&F_{n-k+1}\left(\!\binom{n}{k}\!\right) + F_k\left(\!\binom{n}{k+1}\!\right) \\
&= F_{n-k+1}\frac{F_n\cdots F_{n-k+1}}{F_k\cdots F_1} + F_k\frac{F_n\cdots F_{n-k}}{F_{k+1}\cdots F_1} \\
&= F_{k+1}F_{n-k+1}\frac{F_n\cdots F_{n-k+1}}{F_{k+1}\cdots F_1} + F_kF_{n-k}\frac{F_n\cdots F_{n-k+1}}{F_{k+1}\cdots F_1} \\
&= (F_{k+1}F_{n-k+1} + F_kF_{n-k})\frac{F_n\cdots F_{n-k+1}}{F_{k+1}\cdots F_1} \\
&= (F_{n+1})\frac{F_n\cdots F_{n-k+1}}{F_{k+1}\cdots F_1} \;=\; \left(\!\binom{n+1}{k+1}\!\right).
\end{aligned}$$

In the fourth line we used the identity $F_aF_b + F_{a+1}F_{b+1} = F_{a+b+1}$ with $a = k$ and $b = n - k$. These steps are valid as long as $n > k$ (so that $\left(\!\binom{n}{k+1}\!\right)$ is defined) and $k > 0$ (so that we can write $F_k \cdots F_1$). When $k = 0$ the identity reduces to just $F_{n+1} = F_{n+1}$, which is obviously true.

Assuming that all the entries in one row of the Fibonomial triangle are integers, we can now deduce that all the entries in the next row are also. The above identity ensures that $\left(\!\binom{n+1}{k+1}\!\right)$ is an integer, except that it doesn't apply to $\left(\!\binom{n+1}{0}\!\right)$, $\left(\!\binom{n+1}{1}\!\right)$ or $\left(\!\binom{n+1}{n+1}\!\right)$ since we must have $n > k > 0$. But it is routine to check that these Fibonomials have values 1, F_{n+1} and 1, which are still integers.

Part vi: The following statements are all equivalent to the original one:

$$\begin{aligned} F_{n+1}^2 - F_{n-1}^2 &= 2F_n^2 + F_{n-1}^2 - F_{n-2}^2 \\ (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) &= 2F_n^2 + (F_{n-1} - F_{n-2})(F_{n-1} + F_{n-2}) \\ (F_n)(F_{n+1} + F_{n-1}) &= 2F_n^2 + (F_{n-1} - F_{n-2})(F_n) \\ F_{n+1} + F_{n-1} &= 2F_n + F_{n-1} - F_{n-2} \end{aligned}$$

Now cancelling the F_{n-1} term and using $F_n - F_{n-2} = F_{n-1}$, we arrive at $F_{n+1} = F_n + F_{n-1}$, which is clearly true.

It appears that the coefficients in these remarkable identities are Fibonomial numbers! Following the same pattern, we conjecture that

$$F_{n+1}^4 = 5F_n^4 + 15F_{n-1}^4 - 15F_{n-2}^4 - 5F_{n-3}^4 \pm F_{n-4}^4,$$

where it is not clear yet whether we should use a '+' or a '−' for the final term. Testing our conjecture for $n = 5$ yields

$$\begin{aligned} 8^4 &= 5(5^4) + 15(3^4) - 15(2^4) - 5(1^4) \pm (1^4) \\ &= 3125 + 1215 - 240 - 5 \pm 1 \\ &= 4095 \pm 1. \end{aligned}$$

Since $8^4 = 2^{12} = 4096$, it is now clear that we should use the '+' sign. In general, it turns out that the plus and minus signs alternate in pairs.

Thanks to Ron Knott for assembling an informative web page from which some of the material on this test was drawn. The interested reader should Google "`Knott fibonomial`" *to visit his page.*

February 2009

The Mandelbrot Team Play

Round Two Solutions

Greater Testing Concepts
PO Box 760
Potsdam, NY 13676

The Mandelbrot Team Play
web.mandelbrot.org
info@mandelbrot.org