Problem. For reals x, y, z, we define $f(x, y, z) = min\{|x - y|, |y - z|, |z - x|\}$. Given that a, b, c are the sides of a triangle with perimeter 1, what is the smallest real number L such that $f(a, b, c) < L$ for all possible values of a, b, c?

Solution. We claim that $L = \frac{1}{6}$. Let $p = a + b - c$, $q = b + c - a$, and $r = c + a - b$. We note that $p + q + r = a + b + c = 1$, and since a, b, c are the sides of a triangle, we know that p, q, r are positive reals less than 1. Since $|p - q| = |2a - 2c| = 2|c - a|$ and etc., we clearly have $f(p, q, r) = 2 \cdot f(a, b, c)$. Therefore, we only need to calculate the upper limit of $2 \cdot f(p, q, r)$, and we no longer have the triangle restriction.

Without loss of generality, let $p \geq q \geq r$, and denote $m = q - r$ and $n = p - q$. Then, $min\{|p - q|, |q - r|, |r - p|\} = min\{m, n, m + n\} = min\{m, n\}$. First assume that $m \leq n$, so $f(p, q, r) = m$. If we substitute $q = r + m$ and $p = q + n = r + m + n$ into $p + q + r = 1$, we obtain $3r + 2m + n = 1$. Since $m < n$, we have $1 = 3r + 2m + n > 3r + 3m$, or $m < \frac{1}{3} - r < \frac{1}{3}$, and thus, $f(p, q, r) < \frac{1}{3}$. Assuming that $n < m$, or $f(p, q, r) = n$, yields the same result.

We have shown that we always have $f(p, q, r) < \frac{1}{3}$, implying that $f(a, b, c) < \frac{1}{6}$. It now suffices to show that $L = \frac{1}{6}$ is the least value with the given property. Let $a = 3d + \frac{\epsilon}{3}$, $b = 2d + \frac{\epsilon}{3}$, and $c = d + \frac{\epsilon}{3}$, where d and ϵ are positive reals such that $a + b + c = 6d + \epsilon = 1$. Clearly, a, b, c could be the sides of a triangle with perimeter 1. Moreover, $f(a, b, c) = min\{|a - b|, |b - c|, |c - a|\} = min\{d, d, 2d\} = d = \frac{1-\epsilon}{6}$. Because the only restriction on ϵ is that it is a positive real, it is clear that no value smaller than $\frac{1}{6}$ will suffice.