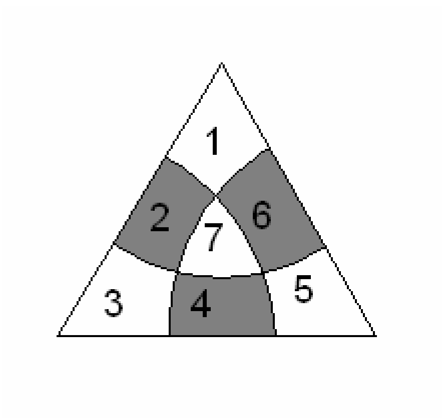
Problem:

A point is randomly chosen inside an equilateral triangle of side length 2. Find the probability that this point is within $\sqrt{2}$ units of exactly two of the triangle's vertices.

Answer: $\boxed{2\sqrt{3}-3}$

Solution:

This is really just a geometry problem that I found was more easily phrased as a probability problem. We want to find the area of the shaded region (shown below) and divide that area by the total area of the triangle.



I numbered the different areas of the triangle to simplify my explanation. Let parentheses denote the area of the specified region, so (1) is equal to the area of the area labeled 1.

Because there is no straight-forward way to determine the area we seek, I will write the areas we can find as a system of equations and then solve for the desired area. These equations may be established by find the area of the entire triangle, the intersection of two of the 60º wedges, and one of the 60º wedges, respectively, which are all basic geometric exercises.

- (1) + (2) + (3) + (4) + (5) + (6) + (7) = $\sqrt{3}$
- (2) + (7) = (4) + (7) = (6) + (7) = $\frac{\pi}{2}-1$ → (2) + (4) + (6) + 3*[(7)] = $\frac{3\pi}{2}-3$
- (1) + (2) + (6) + (7) = (2) + (3) + (4) + (7) = (4) + (5) + (6) + (7) = $\frac{\pi}{3}$

  → (1) + (3) + (5) + 2*[(2) + (4) + (6)] + 3*[(7)] = π

Subtracting two of these equations, we get

[(1) + (3) + (5) + 2*[(2) + (4) + (6)] + 3*[(7)]] – [(1) + (2) + (3) + (4) + (5) + (6) + (7)]
= (2) + (4) + (6) + 2*[(7)] = $\pi - \sqrt{3}$

But we know that (2) + (4) + (6) + 3*[(7)] = $\frac{3\pi}{2}-3$ from above, so subtracting these equations, we find that (7) = $\frac{\pi}{2}-3+\sqrt{3}$

Substituting for (7) in the above equation, we find that (2) + (4) + (6) = $6-3\sqrt{3}$ (I found it interesting that this expression did not contain pi). Anyways, to find the probability, we just divide this by the area of the entire triangle ($\sqrt{3}$) to find that the probability of the randomly chosen point lies within $\sqrt{2}$ of exactly two of the vertices to be $\boxed{2\sqrt{3}-3}$.